

# Killing Tensors from Conformal Killing Vectors

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Some years ago Koutras presented a method of constructing a conformal Killing tensor from a pair of orthogonal conformal Killing vectors. When the vector associated with the conformal Killing tensor is a gradient, a Killing tensor (in general irreducible) can then be constructed. In this paper it is shown that the severe restriction of orthogonality is unnecessary and thus it is possible that many more Killing tensors can be constructed in this way. We also extend, and in one case correct, some results on Killing tensors constructed from a single conformal Killing vector. Weir's result that, for flat space, there are 84 independent conformal Killing tensors, all of which are reducible, is extended to conformally flat spacetimes. In conformally flat spacetimes it is thus possible to construct all the conformal Killing tensors and in particular all the Killing tensors (which in general will not be reducible) from conformal Killing vectors.

## 1 Introduction

In an  $n$ -dimensional Riemannian manifold a *Killing tensor* (of order 2) is a symmetric tensor  $K_{ab}$  satisfying

$$K_{(ab;c)} = 0 \quad (1)$$

A *conformal Killing tensor* (of order two) is a symmetric tensor  $Q_{ab}$  satisfying

$$Q_{(ab;c)} = q_{(a}g_{b)c} \quad \text{with} \quad q_a = (Q_{,a} + 2Q^d_{a;d})/(n+2) \quad (2)$$

where  $Q = Q^d_d$ . In this paper only Killing and conformal Killing tensors of order two will be considered so in future this qualification will be assumed tacitly. When the associated *conformal vector*  $q_a \neq 0$ , the conformal Killing tensor will be called *proper* and otherwise it is a (ordinary) Killing tensor. If  $q_a$  is a Killing vector,  $Q_{ab}$  is referred to as a *homothetic Killing tensor* (see [1] for a discussion of such tensors). If the associated conformal vector  $q_a = q_{,a}$  is the gradient of some scalar field  $q$ , then  $Q_{ab}$  is called a *gradient conformal Killing tensor*. It is easy to see that for each

gradient conformal Killing tensor  $Q_{ab}$  there is an associated Killing tensor  $K_{ab}$  given by

$$K_{ab} = Q_{ab} - qg_{ab} \quad (3)$$

Such a Killing tensor is, of course, only defined up to the addition of a constant multiple of the metric tensor.

Some authors (for example [2, 3]) define a conformal Killing tensor as a *trace-free* tensor  $P_{ab}$  satisfying  $P_{(ab;c)} = p_{(a}g_{bc)}$ . In fact there is no real contradiction or ambiguity between the two definitions. If  $P_{ab}$  is a trace-free conformal Killing tensor then for any scalar field  $\lambda$ ,  $P_{ab} + \lambda g_{ab}$  is a conformal Killing tensor and conversely if  $Q_{ab}$  is a conformal Killing tensor, its trace-free part  $Q_{ab} - \frac{1}{n}Qg_{ab}$  is a trace-free Killing tensor. In this paper the more general definition (2) will be preferred and an explicit trace-free qualification will be added when it is important to distinguish between the two definitions.

Killing tensors are of importance owing to their connection with quadratic first integrals of the geodesic equations: if  $p^a$  is tangent to an affinely parameterised geodesic (i.e.  $p^a{}_{;b}p^b = 0$ ) it is easy to see that  $K_{ab}p^ap^b$  is constant along the geodesic. For *conformal* Killing tensors  $Q_{ab}p^ap^b$  is constant along *null* geodesics and here, of course, only the trace-free part of  $Q_{ab}$  contributes to the constants of motion. Both Killing tensors and conformal Killing tensors are also of importance in connection with the separability of the Hamilton-Jacobi equations and other partial differential equations. Separability will not be considered further here; instead the reader is referred to the literature on this subject (e.g. [4, 5] and the review article [6]).

A Killing tensor is said to be *reducible* if it can be written as a constant linear combination of the metric and symmetrised products of Killing vectors, i.e.

$$K_{ab} = a_0g_{ab} + \sum_{I=1}^N \sum_{J=I}^N a_{IJ}\xi_{I(a}\xi_{|J|b)} \quad (4)$$

where  $\xi_I$  for  $I = 1 \dots N$  are the Killing vectors admitted by the manifold and  $a_0$  and  $a_{IJ}$  for  $1 \leq I \leq J \leq N$  are constants. Generally one is interested only in Killing tensors which are not reducible since the quadratic constant of motion associated with a reducible Killing tensor is simply a constant linear combination of  $p^ap_a$  and of pairwise products of the linear constants of motion  $\xi_{Ia}p^a$ .

## 2 Reducible Conformal Killing Tensors

If  $\chi_1$  and  $\chi_2$  are independent conformal Killing vectors with conformal factors  $\vartheta_1$  and  $\vartheta_2$  (i.e.  $\chi_{1(a;b)} = \vartheta_1g_{ab}$  and  $\chi_{2(a;b)} = \vartheta_2g_{ab}$ ), it is easy to show

$$\chi_{1a}\chi_{1b} \quad \text{and} \quad \chi_{1(a}\chi_{|2|b)} \quad (5)$$

are conformal Killing tensors with associated conformal vectors

$$q_a = \vartheta_1 \chi_{1a} \quad \text{and} \quad q_a = (\vartheta_2 \chi_{1a} + \vartheta_1 \chi_{2a})/2 \quad (6)$$

Clearly the trace-free parts of these two tensors, namely  $\chi_{1a}\chi_{1b} - \frac{1}{n}\chi_1^c\chi_{1c}g_{ab}$  and  $\chi_{1(a}\chi_{2|b)} - \frac{1}{n}\chi_1^c\chi_{2c}g_{ab}$  are trace-free conformal Killing tensors. Note that if  $\chi_1$  is a proper conformal Killing vector, both the conformal Killing tensors in (5) are proper. Note also that if  $\chi_1$  is a homothetic Killing vector with constant conformal factor  $\vartheta_1$  and  $\chi_2$  is a Killing vector, the second conformal Killing tensor in (5) is homothetic.

Koutras [2] proved that if  $\chi_1$  and  $\chi_2$  were *orthogonal* conformal Killing vectors, then  $Q_{ab} = \chi_{1a}\chi_{2b} + \chi_{1b}\chi_{2a}$  was a *trace-free* conformal Killing tensor and that, if  $\chi$  was a *null* conformal Killing vector, then  $Q_{ab} = \chi_a\chi_b$  was a *trace-free* conformal Killing tensor. However, from the above considerations it is clear that the assumptions of orthogonality or nullness are unnecessary; one can simply take the symmetrised product of *any two* conformal Killing vectors to obtain a conformal Killing tensor and then, if a trace-free conformal Killing tensor is required, take the trace-free part of this tensor.

It is easy to see that any scalar multiple of the metric tensor  $\lambda g_{ab}$  is a conformal Killing tensor with associated conformal vector  $q_a = \lambda_{,a}$ . A conformal Killing tensor  $Q_{ab}$  is said to be *reducible* if it can be written as a linear combination of the metric and symmetrised products of conformal Killing vectors:

$$Q_{ab} = \lambda g_{ab} + \sum_{I=1}^N \sum_{J=I}^M a_{IJ} \chi_{I(a} \chi_{J|b)} \quad (7)$$

where  $\chi_I$ , for  $I = 1 \dots M$ , are the conformal Killing vectors admitted by the manifold and  $a_{IJ}$  for  $1 \leq I \leq J \leq M$  are constants. Note that here, unlike in the definition of reducibility in (4), the coefficient  $\lambda$  multiplying the metric is not constant. The conformal vector  $q_a$  associated with the conformal Killing tensor (7) is

$$q_a = \lambda_{,a} + \sum_{I=1}^M \sum_{J=I}^M \frac{1}{2} a_{IJ} (\vartheta_I \chi_{Ja} + \vartheta_J \chi_{Ia}) \quad (8)$$

Thus, if the conformal Killing vectors of a manifold are known, the following prescription for finding Killing tensors suggests itself: construct all the reducible conformal Killing tensors as in (7) and then determine which of these (if any) are gradient (by finding the linear subspace for which  $q_{[a,b]} = 0$ ) and, for this subspace, construct the associated Killing tensors as in (3). Note that if  $Q_{ab}$  is a gradient conformal Killing tensor, then so is  $Q_{ab} + \mu g_{ab}$  for any scalar field  $\mu$ . Thus, without loss of generality in this construction,  $\lambda$  in (7) may be assumed to be zero.

If the metric admits  $N$  independent Killing vectors  $\xi_1, \dots, \xi_N$ , the linear space of all reducible conformal Killing tensors given by (7) contains a linear subspace of

reducible Killing tensors of the form (4). Since reducible Killing tensors are of little interest, these reducible tensors can be excluded if a basis of the conformal Killing vectors is chosen to be of the form  $\xi_1, \dots, \xi_N, \chi_{N+1}, \dots, \chi_M$  and then only reducible conformal Killing tensors of the following form are considered:

$$Q_{ab} = \sum_{I=1}^N \sum_{J=N+1}^M a_{IJ} \xi_{I(a} \chi_{|J|b)} + \sum_{I=N+1}^M \sum_{J=I}^M a_{IJ} \chi_{I(a} \chi_{|J|b)} \quad (9)$$

Occasionally the Killing tensors constructed in this way will be reducible and in general it is necessary to check whether they can be expressed in the form (4).

### 3 Killing Tensors from 1 Conformal Killing Vector

**Theorem 1.** Any manifold which admits a proper non-null conformal Killing vector field  $\chi$  which is geodesic (that is  $\chi_{a;b} \chi^b = \lambda \chi_a$ ) also admits the Killing tensor  $K_{ab} = \chi_a \chi_b - \frac{1}{2} \chi^2 g_{ab}$ , where  $\chi^2 = \chi^c \chi_c$ .

**Proof.** Contracting the equation  $\chi_{(a;b)} = \vartheta g_{ab}$  with  $\chi^a \chi^b$  gives  $\lambda \chi^2 = \vartheta \chi^2$ . Hence as  $\chi$  is non-null,  $\lambda = \vartheta$ . Contracting  $\chi_{(a;b)} = \vartheta g_{ab}$  with  $\chi^b$  gives  $\vartheta \chi_a = \frac{1}{2} (\chi^2)_{,a}$  and hence from equations (3, 5 & 6),  $K_{ab} = \chi_a \chi_b - \frac{1}{2} \chi^2 g_{ab}$  is a Killing tensor.

Koutras [2] proved this result for *homothetic* Killing vectors only. Our proof is more general and direct as it does not rely on the introduction of a particular co-ordinate system. Koutras claimed that the result was valid for null  $\chi$ , but this is false as the following counter-example shows. Consider the metric:

$$ds^2 = e^{2u} (2A(x, y, v) du dv + dx^2 + dy^2) \quad (10)$$

It is easy to see that  $\chi^a$  is a null homothetic Killing vector with conformal factor  $\vartheta = 1$ . As  $\chi^a$  is a null conformal Killing vector, it is necessarily geodesic. The associated conformal Killing tensor  $Q_{ab}$  and conformal vector  $q_a$  are

$$Q_{ab} = A^2 e^{4u} \delta_a^v \delta_b^v \quad q_a = e^{2u} \delta_a^v \quad (11)$$

A straightforward calculation shows that  $q_a$  is not a gradient.

The following theorem generalises that of Amery & Maharaj [3] for homothetic Killing vectors.

**Theorem 2.** Any manifold which admits a conformal Killing vector  $\chi$  that is a gradient vector, also admits the Killing tensor  $K_{ab} = \chi_a \chi_b - \frac{1}{2} \chi^2 g_{ab}$ .

**Proof.** As  $\chi_a$  is a gradient,  $\chi_{[a;b]} = 0$  and thus  $\chi_{a;b} = \vartheta g_{ab}$  and contraction with  $\chi^b$  gives  $\vartheta \chi_b = \frac{1}{2} (\chi^2)_{,b}$  and the result follows as in the proof of theorem 1.

For non-null vectors theorem 2 also follows from theorem 1 since gradient conformal Killing vectors are geodesic. For the null case a gradient conformal Killing vector is a Killing vector and so the associated Killing tensor is necessarily reducible.

## 4 Conformal Transformations

If  $\chi^a$  is a conformal Killing vector of the metric  $g_{ab}$  with conformal factor  $\vartheta$ , it is also a conformal Killing vector of the metric  $\tilde{g}_{ab} = e^{2\Omega}g_{ab}$  with conformal factor  $\tilde{\vartheta} = \vartheta + \Omega_{,c}\chi^c$ . The analogous result for conformal Killing tensors is:

**Theorem 3.** If  $Q^{ab}$  is a conformal Killing tensor satisfying  $\nabla^{(a}Q^{bc)} = q^{(a}g^{bc)}$ , then  $Q^{ab}$  is also a conformal Killing tensor of the conformally related metric  $\tilde{g}_{ab} = e^{2\Omega}g_{ab}$ .  $Q^{ab}$  satisfies  $\tilde{\nabla}^{(a}Q^{bc)} = \tilde{q}^{(a}\tilde{g}^{bc)}$ , where  $\tilde{q}^a = q^a + 2\Omega_{,d}Q^{da}$ .

**Proof.** The proof is straightforward involving an evaluation of  $\tilde{\nabla}^{(a}Q^{bc)}$  using

$$\tilde{\Gamma}_{bc}^a = \Gamma_{bc}^a + \delta_b^a\Omega_{,c} + \delta_c^a\Omega_{,b} - \Omega^{,a}g_{bc} \quad (12)$$

From this theorem and the corresponding theorem for conformal Killing vectors it follows that the number of linearly independent *trace-free* conformal Killing tensors is invariant under conformal change of metric and the number of linearly independent *reducible* trace-free conformal Killing tensors is similarly invariant. Note the trace-free qualification here; owing to the freedom to add arbitrary multiples of the metric, the number of independent conformal Killing tensors in a manifold is infinite.

The maximal number of independent trace-free conformal Killing tensors admitted by an  $n$ -dimensional Riemannian manifold is  $\frac{1}{12}(n-1)(n+2)(n+3)(n+4)$  and this limit is achieved when the metric is flat [7]. In this case all the conformal Killing tensors are reducible. Noting theorem 3, we see that these results are valid for any conformally flat metric. Thus, in particular, all Killing tensors in a conformally flat spacetime may be constructed from products of conformal Killing vectors.

Amery & Maharaj [3] used Koutras' methods to find Killing tensors in Robertson-Walker metrics (which are conformally flat). They were able to construct only 39 of the 84 possible reducible conformal Killing tensors as they used only mutually orthogonal conformal Killing vectors in their construction. Thus their results are likely to be incomplete. Currently work is in progress investigating Killing tensors in conformally flat spacetimes and the results will be presented elsewhere.

## References

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